

Bivariate Binomial Moments and Bonferroni-type Inequalities*

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Abstract

We obtain bivariate forms of Gumbel's, Fréchet's and Chung's linear inequalities for $P(S \geq u, T \geq v)$ in terms of the bivariate binomial moments $\{S_{i,j}\}$, $1 \leq i \leq k, 1 \leq j \leq l$ of the joint distribution of (S, T) . At $u = v = 1$, the Gumbel and Fréchet bounds improve monotonically with non-decreasing (k, l) . The method of proof uses combinatorial identities, and reveals a multiplicative structure before taking expectation over sample points.

Keywords: bivariate binomial moments, Gumbel's inequality, combinatorial identity, Bonferroni-type inequalities

1 Introduction

The paper of Hoppe and Seneta (2012) gives a simplified treatment of several well-known, and some less well-known bounds on the probability of a union of events. The treatment uses binomial moments of a general non-negative integer-valued random variable T on sample space $\{0, 1, 2, \dots, n\}$ to derive inequalities for $P(T \geq v)$. Gumbel's Identity provides the link to the events setting, where T is the number out of n events which occur.

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The present paper extends the univariate methodology of Hoppe and Seneta (2012) to derive extensions to a bivariate setting of the nature of such inequalities, in particular, of Fréchet, Gumbel and Chung, and of their special univariate properties. The focus is identities and inequalities for $P(S \geq u, T \geq v)$ for a pair of non-negative integer-valued random variables (S, T) on $\{0, 1, 2, \dots, m\} \times \{1, 2, \dots, n\}$.

Suppose $\{A_1, A_2, \dots, A_m\}$ are arbitrary events in a probability space (Ω, B, P) , and

$$S_k = \sum_{i \in I_{k,m}} P(A_{i_1}, A_{i_2}, \dots, A_{i_k}), k \geq 1, \quad (1)$$

with $S_0 = 1$ by definition, where the set $I_{k,m}$ consists of all k -tuples $i = \{i_1, i_2, \dots, i_k\}$, where $1 \leq i_1 < i_2 < \dots < i_k \leq m$. The quantities (1) are called (univariate) Bonferroni Sums. If $S(\omega) = \#\{j : 1 \leq j \leq m, \omega \in A_j\}$ is the number of events A_1, A_2, \dots, A_m which occur at a sample point ω , *Gumbel's Identity* (Gumbel, 1936, 1937) expresses S_k as the k -th binomial moment of the random variables S :

$$S_k = E \binom{S}{k}, \quad (2)$$

Approaches to the use of Bonferroni sums (1) for equalities and inequalities on $P(S = u)$ and $P(S \geq u)$ have from the earliest times (see for example Fréchet (1940), and Galambos and Simonelli (1996)), been imbedded to a greater or lesser extent, in the “events” setting, using in this context manipulations of probabilities of unions, intersections and complements of events.

The total separation of expression of equalities and inequalities in terms of binomial moments using identities in binomial coefficients, from the subsequent application to the “events” setting and expression in terms of Bonferroni sums in that setting, seems arguably a more foundational and direct approach. The derivation of known univariate inequalities, both linear and quadratic, in this manner is presented in Hoppe and Seneta (2012). For example the inequality derived in Hoppe and Seneta (2012), §3.4,

$$P(S \geq 1) \leq \frac{E\left\{\binom{m}{k} - \binom{m-S}{k}\right\}}{\binom{m-1}{k-1}} = \frac{\binom{m}{k} - \overline{S}_k}{\binom{m-1}{k-1}}, k \geq 1 \quad (3)$$

is the expression in terms of binomial moments of Gumbel's (1936) Inequality. Here

$$\overline{S}_k = E \binom{m-S}{k}$$

devolves to $\sum_{i \in I_{k,m}} P(\bar{A}_{i_1}, \bar{A}_{i_2}, \dots, \bar{A}_{i_k})$ in the “events” setting, and hence the numerator of (3) to $\sum_{i \in I_{k,m}} P(A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k})$.

The *bivariate analogue of Gumbel’s Identity* is

$$S_{k,l} = E \binom{S}{k} \binom{T}{l}, \quad (4)$$

where “T” is the counting random variate for the events set B_1, B_2, \dots, B_n and

$$S_{k,l} = \Sigma' P(A_{i_1}, A_{i_2}, \dots, A_{i_k}; B_{j_1}, B_{j_2}, \dots, B_{j_l}), \quad (5)$$

where $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$ are two sets of events in a probability space (Ω, B, P) , the summation Σ' is over the index set $\{(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_l), 1 \leq i_1 < i_2 < \dots < i_k \leq m, 1 \leq j_1 < j_2 < \dots < j_l \leq n\}$.

A general form of Gumbel’s Identity for any finite number of counting random variables $\{S, T, \dots\}$ is proved by the method of Indicators in Galambos and Simonelli (1996), Chapter V, Section VI, so we have (4) to hand, in application of our bivariate binomial moment theory, developed in terms of $S_{k,l}$ for *general* random variables (S, T) , to the “two sets” context.

Our “events-free” methodology and representation in terms of a bivariate distribution (S, T) is not more “general” than the “events” setting, since given such a bivariate distribution it is always possible to construct a probability space and event sets, $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$ in it, for which S, T are the “counting” variables.

The “events” setting is primary in practical applications in calculating *values* of the bivariate binomial moments $S_{i,j}$ for small (i, j) via Bonferroni sums using the the generalized Gumbel Identity, when the bivariate distribution of S, T is not known.

Our methodology via combinatorial identities provides relatively simple proofs, and indicates direction of extension to more than two dimensions.

In the bivariate “events” setting, the initial fundamental results, including (4), for $P(S = u, T = v)$, $P(S \geq u, T \geq v)$ are in the note of Meyer (1969), which leans on an earlier such result in Fréchet (1940), which itself has a convoluted history (see our §2). The work of Chen and Seneta (1996) (2000) (see also the recent book of Chen (2014)) uses to a large extent combinatorial identities, but nevertheless relies on Meyer’s (1969) results to express the bounds in terms of the Bonferroni-type sums (5).

Thus our first task in Section 2 of this paper is to generalize in a self-contained fashion the results of Meyer (1969) for a pair of non-negative integer-valued random variables (S, T) on $\{0, 1, 2, \dots, m\} \times \{0, 1, 2, \dots, n\}$, using (4) as the *definition* of the (k, l) binomial moment of the distribution. We shall need some of these results in the following sections.

2 Basic Identities in Terms of Bivariate Binomial Moments

2.1 The Fundamental Theorem.

Theorem 1. *If (S, T) is a pair of random variables on $\{0, 1, \dots, m\} \times \{0, 1, \dots, n\}$, write $P_{[u,v]} = P(S = u, T = v)$. Then for $m, n \geq u, v \geq 0$,*

$$P_{[u,v]} = P(S = u, T = v) = \sum_{t=u+v}^{m+n} \sum_{i+j=t} (-1)^{t-(u+v)} \binom{i}{u} \binom{j}{v} S_{i,j}, \quad (6)$$

where

$$S_{i,j} = E \binom{S}{i} \binom{T}{j}, m, n \geq i, j \geq 0. \quad (7)$$

Proof. Write for the ordinary bivariate probability generating function:

$$P(t, s) = \sum_{u=0}^m \sum_{v=0}^n t^u s^v P_{[u,v]} \quad (8)$$

Then, eventually using (7):

$$\begin{aligned} P(1+t, 1+s) &= \sum_{u=0}^m \sum_{v=0}^n (1+t)^u (1+s)^v P_{[u,v]} \\ &= \sum_{u=0}^m \sum_{v=0}^n \sum_{i=0}^u \sum_{j=0}^v \binom{u}{i} \binom{v}{j} t^i s^j P_{[u,v]} \\ &= \sum_{i=0}^m \sum_{j=0}^n \sum_{u=i}^m \sum_{v=j}^n \binom{u}{i} \binom{v}{j} t^i s^j P_{[u,v]} \\ &= \sum_{i=0}^m \sum_{j=0}^n t^i s^j S_{i,j}, \end{aligned}$$

so that

$$\begin{aligned}
P(t, s) &= \sum_{i=0}^m \sum_{j=0}^n (t-1)^i (s-1)^j S_{i,j} \\
&= \sum_{i=0}^m \sum_{j=0}^n \sum_{u=0}^i \sum_{v=0}^j \binom{i}{u} t^u (-1)^{i-u} \binom{j}{v} s^v (-1)^{j-v} S_{i,j} \\
&= \sum_{u=0}^m \sum_{v=0}^n \sum_{i=u}^m \sum_{j=v}^n \binom{i}{u} \binom{j}{v} (-1)^{i+j-(u+v)} S_{i,j} t^u s^v.
\end{aligned} \tag{9}$$

By comparing the coefficients of $t^u s^v$ for $P(t, s)$ in (8) and (9), we obtain (6) for $P_{[u,v]}$. \square

This expression (6) may also usefully be written as

$$P_{[u,v]} = \sum_{i=u}^m \sum_{j=v}^n \binom{i}{u} \binom{j}{v} (-1)^{i+j-(u+v)} S_{i,j}, \tag{10}$$

so that, by (7):

Corollary 1. *From (10) for $m, n \geq u, v \geq 0$,*

$$P_{[u,v]} = E \left\{ \left(\sum_{i=u}^m \binom{i}{u} (-1)^{i-u} \binom{S}{i} \right) \left(\sum_{j=v}^n \binom{j}{v} (-1)^{j-v} \binom{T}{j} \right) \right\}. \tag{11}$$

Theorem 1 expresses the bivariate distribution in terms of the bivariate moments $S_{i,j}$.

Our equation (6) is equation (1) in Meyer's (1969) pioneering paper. However, his equation (1) is in terms of the bivariate set-specific quantities (our (5) above), and refers to Fréchet's (1940) booklet for proof in that set-specific context. Fréchet (1940), pp.50-52, obtains the expression using "the second theorem of Broderick", from Broderick (1937), who develops his theory in the "symbolic" fashion akin to Bonferroni's (1936). This symbolic fashion is later present in Chapter 2, Section 6, of the classic book of Riordan (1958), but in terms only of univariate binomial moments.

We thought it time to give a simple, direct and non-symbolic proof as above, which is still in the spirit of Riordan (1958), for completeness, since Theorem 1 is a foundation for Theorem 2 below, whose proof is extremely condensed in Meyer (1969).

However, the following argument for Theorem 1, due partly to Sibuya, and to Galambos and Xu (1995) where Sibuya is acknowledged, is notable because of its development in terms of combinatorial identities:

Since

$$(1-1)^S = \sum_{i=0}^S (-1)^i \binom{S}{i} = \sum_{i=0}^m (-1)^i \binom{S}{i},$$

$$E((1-1)^S) = (1-1)^0 P(S=0) = \sum_{i=0}^m (-1)^i E\binom{S}{i}, \quad (12)$$

where $(1-1)^0$ is interpreted as 1; and similarly for $P(T=0)$.

Now, since $(1-1)^{S+T} = (1-1)^S (1-1)^T$,

$$\begin{aligned} (-1)^t \binom{S+T}{t} &= \sum_{i=0}^t (-1)^i \binom{S}{i} (-1)^{t-i} \binom{T}{t-i} \\ &= \sum_{i+j=t} (-1)^t \binom{S}{i} \binom{T}{j}, \end{aligned}$$

so

$$\sum_{t=0}^{m+n} (-1)^t \binom{S+T}{t} = \sum_{t=0}^{m+n} \sum_{i+j=t} (-1)^t \binom{S}{i} \binom{T}{j},$$

and taking expectation and using (12) with $S+T$ replacing S , we have

$$P(S+T=0) = \sum_{t=0}^{m+n} \sum_{i+j=t} (-1)^t S_{i,j}.$$

Since $P(S+T=0) = P(S=0, T=0)$, we have (6) of our Theorem 1, at $u=v=0$. Theorem 1 for general u and v follows from the special case $u=v=0$ by the reduction method encapsulated in Corollary 2.1 of Galambos and Xu (1995), see also Galambos and Simonelli (1996).

Once univariate *inequalities* are available for $P(S+T=0) = P(S=0, T=0)$, the reduction method leads to inequalities for the general case $P(S=u, T=v)$. The reduction method applied to the classic univariate Bonferroni Inequalities leads to the inequalities of our Theorem 3, which are Meyer's version of extension to the bivariate case.

2.2 Theorems 2 and 3.

The Corollary above, and first part of the proof of Theorem 2 below initiate our methodology of working in terms of identities and inequalities in terms of combinatorial quantities applied to sample points of the pair of random variables $\{S, T\}$ before using the

linear property of expectation to express in terms of bivariate moments of their joint distribution.

Our method of proof is akin to, but distinct from, the method of indicator functions which uses products of indicator functions before taking expectation. The method of indicator functions is concisely exposted at the beginning of Galambos and Xu (1995).

Theorem 2. *Put*

$$P_{(u,v)} = P(S \geq u, T \geq v),$$

then:

$$P_{(u,v)} = \sum_{t=u+v}^{m+n} \sum_{i+j=t} (-1)^{i+j-(u+v)} \binom{i-1}{u-1} \binom{j-1}{v-1} S_{i,j}, \quad (13)$$

where $S_{i,j}$ is given by (7), and, further

$$S_{i,j} = \sum_{u=i}^m \sum_{v=j}^n \binom{u-1}{i-1} \binom{v-1}{j-1} P_{(u,v)}. \quad (14)$$

Proof. From (11) :

$$\begin{aligned} P_{(u,v)} &= \sum_{y=u}^m \sum_{z=v}^n P_{[y,z]} \\ &= \sum_{y=u}^m \sum_{z=v}^n E \left\{ \left(\sum_{i=y}^m \binom{i}{y} (-1)^{i-y} \binom{S}{i} \right) \left(\sum_{j=z}^n \binom{j}{z} (-1)^{j-z} \binom{T}{j} \right) \right\} \\ &= E \left\{ \left(\sum_{y=u}^m \sum_{i=y}^m \binom{i}{y} (-1)^{i-y} \binom{S}{i} \right) \left(\sum_{z=v}^n \sum_{j=z}^n \binom{j}{z} (-1)^{j-z} \binom{T}{j} \right) \right\} \\ &= E \left\{ \left(\sum_{i=u}^m (-1)^{i-u} \binom{i-1}{u-1} \binom{S}{i} \right) \left(\sum_{j=v}^n (-1)^{j-v} \binom{j-1}{v-1} \binom{T}{j} \right) \right\} \end{aligned}$$

where the last line follows from Combinatorial Identity 2 in our Section 6.

Now, the right hand side of (14), using $\binom{d}{k} = \binom{d-1}{k} + \binom{d-1}{k-1}$ (Identity 1, Section 6),

$$\begin{aligned} \sum_{u=i}^m \sum_{v=j}^n \binom{u-1}{i-1} \binom{v-1}{j-1} P_{(u,v)} &= \sum_{u=i}^m \sum_{v=j}^n \left(\binom{u}{i} - \binom{u-1}{i} \right) \left(\binom{v}{j} - \binom{v-1}{j} \right) P_{(u,v)} \\ &= \sum_{u=i}^m \sum_{v=j}^n \binom{u}{i} \binom{v}{j} P_{[u,v]} = S_{i,j}. \end{aligned}$$

□

Theorem 3. *For any non-negative integer k ,*

$$\sum_{t=u+v}^{u+v+2k+1} \sum_{i+j=t} g(i, j; t) \leq P_{(u,v)} \leq \sum_{t=u+v}^{u+v+2k} \sum_{i+j=t} g(i, j; t), \quad (15)$$

where $g(i, j; t) = (-1)^{t-(u+v)} \binom{i-1}{u-1} \binom{j-1}{v-1} S_{i,j}$.

This expresses Bonferroni's Inequalities in our slightly generalized setting of a bivariate distribution of random variables (S, T) and their binomial bivariate moments. This Theorem 3, which generalizes to the bivariate setting Theorem 3 in Hoppe and Seneta (2012), needs no separate proof, since Meyer's (1969) proof is completely appropriate.

The Sobel-Uppuluri-Galambos Inequalities sharpen the Bonferroni Inequalities in the classic single-set context by adding (for the lower bound) and subtracting (for the upper bound) another binomial moment term. These are extended to the context of distribution of a single general integer random variable T on $\{0, 1, \dots, n\}$ by Hoppe and Seneta (2012), Section 3.2. The extension of analogous bounds in this manner for the joint distribution of two general random variables in terms of $S_{k,l}$ is already contained in Chen and Seneta (1996), (2000). Consequently we now proceed to the two-dimensional generalization of the less-known one-dimensional results in Hoppe and Seneta (2012), and do so, as in the one-dimensional case, by using combinatorial methods on sample points.

Our present paper, foreshadowed in Hoppe and Seneta (2012), was also stimulated by the appearance of Mádi-Nagy and Prékopa (2015), which itself is partly motivated by that of Galambos and Xu (1995). We shall say more in our §3.5

3 Generalized Bivariate Fréchet and Gumbel Inequalities

In the generalized univariate setting of Hoppe and Seneta (2012), these inequalities for $P(T \geq 1)$ were expressed in terms of $S_k = E\binom{T}{k}$, $k \geq 0$, and $\overline{S}_k = E\binom{n-T}{k}$.

For the bivariate setting, recall from (4) that $S_{k,l} = E\binom{S}{k}\binom{T}{l}$ is an obvious analogue of $S_k = E\binom{T}{k}$.

The bivariate analogue of $\overline{S}_k = E\binom{n-T}{k}$ is then

$$\overline{S}_{k,l} = \binom{m}{k} E\binom{n-T}{l} + \binom{n}{l} E\binom{m-S}{k} - E\binom{m-S}{k} \binom{n-T}{l}. \quad (16)$$

To see the reason for this, and it is central to our sequel, it is convenient to proceed in terms of the quantity

$$\left\{ \binom{m}{k} - \binom{m-S}{k} \right\} \left\{ \binom{n}{l} - \binom{n-T}{l} \right\} \quad (17)$$

so that, imitating the proof of Fréchet's univariate Inequality in Hoppe and Seneta (2012), §3.3:

$$\begin{aligned} \binom{m}{k} \binom{n}{l} - \overline{S}_{k,l} &= \sum_{i=0}^m \sum_{j=0}^n \left\{ \binom{m}{k} - \binom{m-i}{k} \right\} \left\{ \binom{n}{l} - \binom{n-j}{l} \right\} P_{[i,j]} \\ &\leq \sum_{i=1}^m \sum_{j=1}^n \binom{m}{k} \binom{n}{l} P_{[i,j]} \\ &= \binom{m}{k} \binom{n}{l} P(S \geq 1, T \geq 1). \end{aligned}$$

Thus with the definition (16) of $\overline{S}_{k,l}$ we can write down:

3.1 Fréchet's Bivariate Inequality

$$P(S \geq 1, T \geq 1) \geq \frac{\binom{m}{k} \binom{n}{l} - \overline{S}_{k,l}}{\binom{m}{k} \binom{n}{l}}. \quad (18)$$

Furthermore we have an “expectation of product” form:

$$\binom{m}{k} \binom{n}{l} - \overline{S}_{k,l} = E \left\{ \left\{ \binom{m}{k} - \binom{m-S}{k} \right\} \left\{ \binom{n}{l} - \binom{n-T}{l} \right\} \right\}. \quad (19)$$

Expressions of products of linear forms in binomial coefficients pervade our subsequent development, and are already evident in our §1, for example in (10). They make it clear how our various bivariate equalities and inequalities generalize from univariate, and allow generalization to multivariate.

Before proceeding we note that in (18) above, and all the bounds below involving $\overline{S}_{k,l}$, can be expressed in the customary way in terms of linear combinations of the bivariate moments $S_{i,j}$ on account of (42) in the sequel, according to which:

$$\overline{S}_{k,l} = \binom{m}{k} \binom{n}{l} - \sum_{s=1}^k \sum_{r=1}^l (-1)^{s+r} \binom{m-s}{k-s} \binom{n-r}{l-r} S_{s,r}. \quad (20)$$

3.2 Gumbel's Bivariate Inequality

$$P(S \geq 1, T \geq 1) \leq \frac{\binom{m}{k} \binom{n}{l} - \bar{S}_{k,l}}{\binom{m-1}{k-1} \binom{n-1}{l-1}}. \quad (21)$$

Proof. When $i \geq 1$, $\binom{m-1}{k} \geq \binom{m-i}{k}$, so from Combinatorial Identity 1 (in Section 6),

$$\begin{aligned} \binom{m}{k} - \binom{m-i}{k} &= \binom{m-1}{k} + \binom{m-1}{k-1} - \binom{m-i}{k} \\ &\geq \binom{m-1}{k-1}. \end{aligned}$$

Therefore, from the proof of Fréchet's Bivariate Inequality above,

$$\begin{aligned} \binom{m}{k} \binom{n}{l} - \bar{S}_{k,l} &= \sum_{i=0}^m \sum_{j=0}^n \left\{ \binom{m}{k} - \binom{m-i}{k} \right\} \left\{ \binom{n}{l} - \binom{n-j}{l} \right\} P(S=i, T=j) \\ &\geq \sum_{i=1}^m \sum_{j=1}^n \binom{m-1}{k-1} \binom{n-1}{l-1} P(S=i, T=j) \\ &= \binom{m-1}{k-1} \binom{n-1}{l-1} P(S \geq 1, T \geq 1), \end{aligned}$$

which reduces to (21). \square

Note that the numerator in both cases (18) and (21) is (19), as is the case also in the right hand side of (35) below.

3.3 Monotonicity and Concavity of Fréchet's Bounds

Put

$$L_{k,l} = \frac{\left\{ \binom{m}{k} - \binom{m-S}{k} \right\} \left\{ \binom{n}{l} - \binom{n-T}{l} \right\}}{\binom{m}{k} \binom{n}{l}}. \quad (22)$$

Fix l , then

$$\begin{aligned} L_{k,l} - L_{k+1,l} &= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \left\{ \left(1 - \frac{\binom{m-S}{k}}{\binom{m}{k}} \right) - \left(1 - \frac{\binom{m-S}{k+1}}{\binom{m}{k+1}} \right) \right\} \\ &= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \left\{ \frac{(m-S)!(m-k-1)!}{(m-S-k-1)!m!} - \frac{(m-S)!(m-k)!}{(m-S-k)!m!} \right\} \\ &= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \frac{(m-S)!(m-k-1)!}{(m-S-k-1)!m!} \left(1 - \frac{m-k}{m-S-k} \right). \end{aligned} \quad (23)$$

Since $1 - \frac{m-k}{m-S-k} \leq 0$, we get

$$L_{k,l} \leq L_{k+1,l}. \quad (24)$$

Taking expectations in both sides of (22) and (24), we have Fréchet's bound equal to $E(L_{k,l})$. So Fréchet's bound increases with increasing k . Similarly, Fréchet's bound increases with increasing l .

To prove concavity, use (23),

$$\begin{aligned}
L_{k,l} - L_{k+1,l} &= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \frac{(m-S)!(m-k-1)!}{(m-S-k-1)!m!} \left(1 - \frac{m-k}{m-S-k}\right) \\
&= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \frac{(m-S)!(m-k-2)!}{(m-S-k-2)!m!} \frac{m-k-1}{m-S-k-1} \frac{-S}{m-S-k} \\
&= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \frac{(m-S)!(m-k-2)!}{(m-S-k-2)!m!} \frac{m-k-1}{m-S-k-1} \frac{m-S-k-1}{m-S-k} \frac{-S}{m-S-k-1} \\
&= \frac{\binom{n}{l} - \binom{n-T}{l}}{\binom{n}{l}} \frac{(m-S)!(m-k-2)!}{(m-S-k-2)!m!} \frac{m-k-1}{m-S-k} \left(1 - \frac{m-k-1}{m-S-k-1}\right) \\
&= \frac{m-k-1}{m-S-k} (L_{k+1,l} - L_{k+2,l}).
\end{aligned}$$

Since for $S = 0$, $L_{k,l} - L_{k+1,l} = 0$ for all $k \geq 0$, and for $1 \leq S \leq n$, $\frac{m-k-1}{m-S-k} \geq 1$ and $L_{k,l} - L_{k+1,l} \leq 0$ for all $k \geq 0$, so

$$L_{k,l} - L_{k+1,l} \leq L_{k+1,l} - L_{k+2,l}.$$

Therefore, Fréchet's bounds is concave in k . We can prove the concavity with respect to the second parameter l similarly.

3.4 Monotonicity and Convexity of Gumbel's Bounds

Gumbel's inequality applies equally to random variables $U = m - S$ and $V = n - T$. So,

$$P(U \geq 1, V \geq 1) \leq \frac{E\left\{\binom{m}{k} - \binom{S}{k}\right\}\left\{\binom{n}{l} - \binom{T}{l}\right\}}{\binom{m-1}{k-1}\binom{n-1}{l-1}}. \quad (25)$$

Denote the right hand side of (25) as $G_{k,l}$. By a double use of Combinatorial Identity 3 (Section 6):

$$E\binom{S}{k} = \binom{m}{k} - \sum_{i=k}^m \binom{i-1}{k-1} P(S < i), \quad (26)$$

and from our Theorem 2, specifically (14):

$$E\binom{S}{k}\binom{T}{l} = \sum_{i=k}^m \sum_{j=l}^n \binom{i-1}{k-1} \binom{j-1}{l-1} P(S \geq i, T \geq j), \quad (27)$$

we have

$$\begin{aligned}
& \binom{m}{k} \binom{n}{l} - \binom{m}{k} E \binom{T}{l} - \binom{n}{l} E \binom{S}{k} + E \binom{S}{k} \binom{T}{l} \\
&= \binom{m}{k} \binom{n}{l} - \binom{m}{k} \left\{ \binom{n}{l} - \sum_{j=l}^n \binom{j-1}{l-1} P(T < j) \right\} \\
&- \binom{n}{l} \left\{ \binom{m}{k} - \sum_{i=k}^m \binom{i-1}{k-1} P(S < i) \right\} \\
&+ \sum_{i=k}^m \sum_{j=l}^n \binom{i-1}{k-1} \binom{j-1}{l-1} \{1 - P(S \geq i, T < j) - P(S < i, T \geq j) - P(S < i, T < j)\} \\
&= \sum_{i=k}^m \sum_{j=l}^n \binom{i-1}{k-1} \binom{j-1}{l-1} \{P(T < j) + P(S < i) - P(S \geq i, T < j) \\
&- P(S < i, T \geq j) - P(S < i, T < j)\} \\
&= \sum_{i=k}^m \sum_{j=l}^n \binom{i-1}{k-1} \binom{j-1}{l-1} P(S < i, T < j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
G_{k,l} &= \frac{\sum_{i=k}^m \sum_{j=l}^n \binom{i-1}{k-1} \binom{j-1}{l-1} P(S < i, T < j)}{\binom{m-1}{k-1} \binom{n-1}{l-1}} \\
&= \sum_{j=l}^n \frac{\binom{j-1}{l-1}}{\binom{n-1}{l-1}} \sum_{i=k}^m \frac{(i-1)(i-2)\dots(i-k+1)}{(m-1)(m-2)\dots(m-k+1)} P(S < i, T < j).
\end{aligned}$$

Fix l , then we have

$$G_{k,l} - G_{k+1,l} = \sum_{j=l}^n \frac{\binom{j-1}{l-1}}{\binom{n-1}{l-1}} \sum_{i=k}^m \frac{(i-1)(i-2)\dots(i-k+1)}{(m-1)(m-2)\dots(m-k+1)} \left(1 - \frac{i-k}{m-k}\right) P(S < i, T < j). \quad (28)$$

Since $k \leq i \leq m$, so $G_{k,l} \geq G_{k+1,l}$. Thus, $G_{k,l}$ is decreasing with increasing k . We can get a similar conclusion for the second parameter l .

Put $D_{k,l} = G_{k,l} - G_{k+1,l}$. Using (28),

$$D_{k,l} - D_{k-1,l} = \sum_{j=l}^n \frac{\binom{j-1}{l-1}}{\binom{n-1}{l-1}} \sum_{i=k-1}^m \frac{(i-1)(i-2)\dots(i-k+2)}{(m-1)(m-2)\dots(m-k+1)} (m-i) \left(\frac{i-k+1}{m-k} - 1\right) P(S < i, T < j).$$

Since for $i = m$, $D_{k,l} - D_{k-1,l} = 0$ for all k , and for $k-1 \leq i < m$, $\frac{i-k+1}{m-k} - 1 \leq 0$, so we obtain that $D_{k,l} - D_{k-1,l} \leq 0$ for all k . So, Gumbel's bound is convex in k . We can conclude similarly the convexity of Gumbel's bound with respect to the second parameter l .

3.5 Comparison

For fixed $k, l \geq 1$, the Fréchet's lower bound (18) and Gumbel's upper bound (21) for $P_{(1,1)} = P(S \geq 1, T \geq 1)$, when expressed, using (19), in terms of bivariate moments $S_{i,j}$ are a linear combination of all such moments for $1 \leq i \leq k, 1 \leq j \leq l$. Although we have shown the monotonicity of these bounds with non-decreasing (k, l) , it is appropriate to compare these bounds, for small fixed (k, l) , with existing bounds, some of which are optimal in a linear sense.

The linear upper bound

$$P_{(1,1)} \leq S_{1,1} - \frac{2}{n}S_{1,2} - \frac{2}{m}S_{2,1} + \frac{4}{mn}S_{2,2} \quad (29)$$

due to Galambos and Xu (1993) is optimal in terms of $\{S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}\}$ in several senses (see also Seneta and Chen (1996)).

The bivariate Gumbel upper bound (21) at $k = l = 2$, reads

$$P_{(1,1)} \leq S_{1,1} - \frac{1}{n-1}S_{1,2} - \frac{1}{m-1}S_{2,1} + \frac{1}{(m-1)(n-1)}S_{2,2} \quad (30)$$

is weaker, in an obvious sense, excepts when $m = n = 2$. For the example on p.103 of Chen and Seneta (1995), where $m = n = 3$, the right-hand sides of (29) and (30) are respectively 0.888 and 0.891.

A lower bound (Chen and Seneta (1995)) for $P_{(1,1)}$ analogous to (29) is

$$P(S \geq 1, T \geq 1) \geq \frac{4S_{1,1}}{(a+1)(b+1)} - \frac{4S_{1,2}}{b(a+1)(b+1)} - \frac{4S_{2,1}}{a(a+1)(b+1)} + \frac{4S_{2,2}}{ab(a+1)(b+1)} \quad (31)$$

where a and b are integers, providing $m - 2a - 1 \leq 0, n - 2b - 1 \leq 0$. Thus, if we choose $a = m - 1, b = n - 1$, we obtain

$$P(S \geq 1, T \geq 1) \geq \frac{4}{mn}S_{1,1} - \frac{4}{mn(n-1)}S_{1,2} - \frac{4}{mn(m-1)}S_{2,1} + \frac{4}{mn(m-1)(n-1)}S_{2,2}, \quad (32)$$

which is Fréchet's bivariate lower bound (18).

Note that both (29) and (31) were originally obtained using inequalities for linear functions of combinatorial quantities on bivariate sample points.

Very recently Mádi-Nagy and Prékopa (2015), in somewhat the same tradition as Galambos and Xu (1993), (1995), have looked for coefficients $c_{s,t}, d_{s,t}$, $s+t \leq w$ to satisfy

$$\sum_{s=0}^w \sum_{t=0}^{w-s} c_{s,t} S_{s,t} \leq r(u, v; m, n) \leq \sum_{s=0}^w \sum_{t=0}^{w-s} d_{s,t} S_{s,t} \quad (33)$$

where $w \leq \min(m, n)$, and $r(u, v; m, n) = P(S = u, T = v)$ or $r(u, v; m, n) = P(S \geq u, T \geq v)$. The authors use a linear programming approach for functions defined on the joint sample space of (S, T) to obtain, after taking expectation, linear bounds for $r(u, v; m, n)$. These linear bounds are optimal in a certain sense.

The constraint in (33) that $s + t \leq w$ is justified by the authors (p.25): “usually the probabilities of intersections are given up to a certain number of counts, hence the multivariate moments up to a certain total order can be calculated.” However, this makes a little awkward comparison where the constraint on included bivariate binomial moments is $1 \leq s \leq k, 1 \leq t \leq l$: for example, comparing $w = 4$, with $k = l = 2$. Nevertheless, the upper bound (3.7) of Mádi-Nagy and Prékopa (2015),

$$P_{(1,1)} \leq \min \left(S_{1,1} - \frac{2}{mn} S_{1,2} - \frac{2}{m} S_{2,1}, S_{1,1} - \frac{2}{n} S_{1,2} - \frac{2}{mn} S_{2,1} \right) \quad (34)$$

is clearly better than the bound

$$P_{(1,1)} \leq \min \left(S_{1,1} - \frac{1}{m-1} S_{2,1}, S_{1,1} - \frac{1}{n-1} S_{1,2} \right),$$

obtained from the cases $k = 1, l = 2$ and $k = 2, l = 1$ of the Gumbel upper bound (21).

4 Fréchet-type and Gumbel-type Inequalities

Theorem 4. For $1 \leq s, k \leq m$ and $1 \leq t, l \leq n$,

$$1 - \frac{\overline{S}_{k,l}}{\binom{m-s+1}{k} \binom{n-t+1}{l}} \leq P(S \geq s, T \geq t) \leq \frac{\binom{m}{k} \binom{n}{l} - \overline{S}_{k,l}}{\left(\binom{m}{k} - \binom{m-s}{k} \right) \left(\binom{n}{l} - \binom{n-t}{l} \right)}. \quad (35)$$

Proof. For the lower bound, from (19),

$$\begin{aligned}
& \binom{m}{k} \binom{n}{l} - \overline{S}_{k,l} \\
&= \sum_{i=0}^m \sum_{j=0}^n \left(\binom{m}{k} - \binom{m-i}{k} \right) \left(\binom{n}{l} - \binom{n-j}{l} \right) P_{[i,j]} \\
&\leq \sum_{i=0}^{s-1} \sum_{j=0}^{t-1} \left(\binom{m}{k} - \binom{m-s+1}{k} \right) \left(\binom{n}{l} - \binom{n-t+1}{l} \right) P_{[i,j]} \\
&+ \sum_{i=0}^{s-1} \sum_{j=t}^n \left(\binom{m}{k} - \binom{m-s+1}{k} \right) \binom{n}{l} P_{[i,j]} + \sum_{i=s}^m \sum_{j=0}^{t-1} \binom{m}{k} \left(\binom{n}{l} - \binom{n-t+1}{l} \right) P_{[i,j]} \\
&+ \sum_{i=s}^m \sum_{j=t}^n \binom{m}{k} \binom{n}{l} P_{[i,j]} \\
&\leq \binom{m}{k} \binom{n}{l} - \sum_{i=0}^{s-1} \sum_{j=0}^{t-1} \left[\binom{m-s+1}{k} \binom{n}{l} + \binom{m}{k} \binom{n-t+1}{l} \right. \\
&- \left. \binom{m-s+1}{k} \binom{n-t+1}{l} \right] P_{[i,j]} - \sum_{i=0}^{s-1} \sum_{j=t}^n \binom{m-s+1}{k} \binom{n-t+1}{l} P_{[i,j]} \\
&- \sum_{i=s}^m \sum_{j=0}^{t-1} \binom{m-s+1}{k} \binom{n-t+1}{l} P_{[i,j]} \\
&\leq \binom{m}{k} \binom{n}{l} - \sum_{i=0}^{s-1} \sum_{j=0}^{t-1} \binom{m-s+1}{k} \binom{n-t+1}{l} P_{[i,j]} \\
&- \sum_{i=0}^{s-1} \sum_{j=t}^n \binom{m-s+1}{k} \binom{n-t+1}{l} P_{[i,j]} - \sum_{i=s}^m \sum_{j=0}^{t-1} \binom{m-s+1}{k} \binom{n-t+1}{l} P_{[i,j]} \\
&= \binom{m}{k} \binom{n}{l} - \binom{m-s+1}{k} \binom{n-t+1}{l} (1 - P(S \geq s, T \geq t)),
\end{aligned}$$

which reduces to the left hand side of (35).

Next, again using (19) as initial step,

$$\begin{aligned}
\binom{m}{k} \binom{n}{l} - \overline{S}_{k,l} &\geq \sum_{i=s}^m \sum_{j=t}^n \left(\binom{m}{k} - \binom{m-i}{k} \right) \left(\binom{n}{l} - \binom{n-j}{l} \right) P_{[i,j]} \\
&\geq \sum_{i=s}^m \sum_{j=t}^n \left(\binom{m}{k} - \binom{m-s}{k} \right) \left(\binom{n}{l} - \binom{n-t}{l} \right) P_{[i,j]} \\
&= \left(\binom{m}{k} - \binom{m-s}{k} \right) \left(\binom{n}{l} - \binom{n-t}{l} \right) P(S \geq s, T \geq t),
\end{aligned}$$

which reduces to the right hand side of (35). \square

5 Monotonicity and Convexity in Chung-type Bounds

Define

$$A_{k,l}^{(s,t)} = \frac{\sum_{i=s}^k \sum_{j=t}^l (-1)^{i+j-(s+t)} \binom{i-1}{i-s} \binom{m-i}{k-i} \binom{j-1}{j-t} \binom{n-j}{l-j} S_{i,j}}{\binom{m-s}{k-s} \binom{n-t}{l-t}}. \quad (36)$$

We follow Hoppe and Seneta (2012), p.283, from their equation (43), to prove that $A_{k,l}^{(s,t)}$ is monotone and convex in k and l . For $k = s, s+1, \dots, m-1$,

$$\begin{aligned} A_{k,l}^{(s,t)} - A_{k+1,l}^{(s,t)} &= \frac{\sum_{j=t}^l (-1)^{j-t} \binom{j-1}{j-t} \binom{n-j}{l-j}}{\binom{n-t}{l-t}} \sum_{i=s}^{k+1} (-1)^{i-s} \binom{i-1}{i-s} \left[\frac{\binom{m-i}{k-i}}{\binom{m-s}{k-s}} - \frac{\binom{m-i}{k+1-i}}{\binom{m-s}{k+1-s}} \right] S_{i,j} \\ &= \frac{\sum_{j=t}^l (-1)^{j-t} \binom{j-1}{j-t} \binom{n-j}{l-j}}{\binom{n-t}{l-t}} \sum_{i=s}^{k+1} (-1)^{i-s-1} \frac{s}{m-s} \frac{\binom{i-1}{i-s-1} \binom{m-i}{k+1-i}}{\binom{m-s-1}{k-s}} S_{i,j} \\ &= \frac{s}{m-s} A_{k+1,l}^{(s+1,t)} \geq 0. \end{aligned}$$

Therefore, using monotonicity in k , we have

$$A_{k,l}^{(s,t)} - A_{k+1,l}^{(s,t)} = \frac{s}{m-s} A_{k+1,l}^{(s+1,t)} \geq \frac{s}{m-s} A_{k+2,l}^{(s+1,t)} = A_{k+1,l}^{(s,t)} - A_{k+2,l}^{(s,t)}$$

proving convexity in k . Analogous results in l follow similarly.

Again, the proof above is essentially the univariate one from Hoppe and Seneta (2012), p. 283, because of the product structure of (36), before taking expectation:

$$A_{k,l}^{(s,t)} = E \left\{ \frac{\sum_{i=s}^k (-1)^{i-s} \binom{i-1}{i-s} \binom{m-i}{k-i} \binom{S}{i}}{\binom{m-s}{k-s}} \frac{\sum_{j=t}^l (-1)^{j-t} \binom{j-1}{j-t} \binom{n-j}{l-j} \binom{T}{j}}{\binom{n-t}{l-t}} \right\}. \quad (37)$$

When $k = m$ and $l = n$, we see from Theorem 2, and specifically from (13), that $A_{m,n}^{(s,t)} = P(S \geq s, T \geq t)$, and since, for example, $A_{m-1,n}^{(s,t)} \geq A_{m,n}^{(s,t)}$, by monotonicity, we have

$$P(S \geq s, T \geq t) \leq A_{m-1,n}^{(s,t)}. \quad (38)$$

So such (Chung-type) upper bounds are of the nature of the Gumbel-type upper bounds as on the right-hand side of (35). Note that the bound in (38) is already expressed as a linear combination of bivariate binomial moments $S_{i,j}$.

6 Combinatorial Identities

Note that here and throughout this paper, for any real number d and integer $r > 0$, $\binom{d}{r} = \frac{d(d-1)\dots(d-r+1)}{r!}$. If $r = 0$, $\binom{d}{r} = 1$. If m and r are positive integers, and $r > m$, then $\binom{m}{r} = 0$.

Identities 1 to 4 occur, with the same numbering, and are proved, in Hoppe and Seneta (2012), p. 273. They are stated here for the reader's convenience, since all are used in this paper.

Identity 1 (Extended Pascal's Identity) For any real number d and integer $k \geq 1$,

$$\binom{d}{k} = \binom{d-1}{k} + \binom{d-1}{k-1}.$$

Identity 2 For integers $n \geq 1, k \geq 0$,

$$\sum_{x=0}^k (-1)^x \binom{n}{x} = (-1)^k \binom{n-1}{k}.$$

Identity 3 For $k \geq 1, n, k$ integers

$$\binom{n}{k} = \sum_{x=k}^n \binom{x-1}{k-1} = \sum_{x=0}^n \binom{x-1}{k-1}.$$

Identity 4 For $n \geq k \geq 1, r \geq 1$,

$$\binom{n}{k} = \sum_{j=1}^{r-1} \binom{n-j}{k-1} + \binom{n-r+1}{k}.$$

Identity 5

$$\binom{n-T}{l} = \sum_{r=0}^l (-1)^r \binom{n-r}{l-r} \binom{T}{r}, T = 0, 1, 2, \dots, n. \quad (39)$$

Proof. When $l = 0, 1$, we see that (39) holds for every $n \geq l$. Assume when $l = k$, (39) holds for every $n \geq l$. By this induction assumption, it holds for $n-1, n-2, \dots, k+1, k$.

Using Identity 1, iterating back,

$$\begin{aligned} \binom{n-T}{k+1} &= \binom{n-T-1}{k} + \binom{n-T-1}{k+1} \\ &= \binom{n-T-1}{k} + \binom{n-T-2}{k} + \dots + \binom{k+1}{k} + \binom{k+1}{k+1} \\ &= \sum_{N=T}^{n-1} \sum_{r=0}^k (-1)^r \binom{N-r}{k-r} \binom{T}{r} \end{aligned}$$

by the induction hypothesis; and now, after an exchange of order of summation, by Identity 4:

$$\begin{aligned}
&= \sum_{r=0}^k (-1)^r \sum_{N=T}^{n-1} \binom{N-r}{k-r} \binom{T}{r} \\
&= \sum_{r=0}^k (-1)^r \binom{n-r}{k+1-r} \binom{T}{r} + \sum_{r=0}^k (-1)^{r+1} \binom{T-r}{k+1-r} \binom{T}{r} \\
&= \sum_{r=0}^k (-1)^r \binom{n-r}{k+1-r} \binom{T}{r} + \sum_{r=0}^k (-1)^{r+1} \binom{T}{k+1} \binom{k+1}{r}.
\end{aligned}$$

So the only thing we need to prove now is that

$$\sum_{r=0}^k (-1)^{r+1} \binom{T}{k+1} \binom{k+1}{r} = (-1)^{k+1} \binom{T}{k+1}.$$

From Identity 2 above,

$$\sum_{r=0}^k (-1)^{r+1} \binom{k+1}{r} = (-1)^{k+1}.$$

So we have the desired conclusion, by induction. \square

We now apply Identity 5, to express $\overline{S}_l, \overline{S}_{k,l}$ as linear functions of binomial moments.

1. Univariate case

Taking expectations in both sides of (39), we have that,

$$E \binom{n-T}{l} = \sum_{r=0}^l (-1)^r \binom{n-r}{l-r} E \binom{T}{r},$$

which is

$$\overline{S}_l = \sum_{r=0}^l (-1)^r \binom{n-r}{l-r} S_r. \tag{40}$$

2. Bivariate case

Using (39), we have

$$\binom{m-S}{k} \binom{n-T}{l} = \sum_{s=0}^k \sum_{r=0}^l (-1)^{s+r} \binom{m-s}{k-s} \binom{n-r}{l-r} \binom{S}{s} \binom{T}{r}. \tag{41}$$

Take expectations in both sides, we have

$$E \binom{m-S}{k} \binom{n-T}{l} = \sum_{s=0}^k \sum_{r=0}^l (-1)^{s+r} \binom{m-s}{k-s} \binom{n-r}{l-r} S_{s,r}.$$

Thus,

$$\begin{aligned}
\overline{S}_{k,l} &= E\binom{m-S}{k}\binom{n}{l} + E\binom{n-T}{l}\binom{m}{k} - E\binom{m-S}{k}\binom{n-T}{l} \\
&= \sum_{s=0}^k (-1)^s \binom{m-s}{k-s} \binom{n}{l} S_s + \sum_{r=0}^l (-1)^r \binom{n-r}{l-r} \binom{m}{k} S_r \\
&\quad - \sum_{s=0}^k \sum_{r=0}^l (-1)^{s+r} \binom{m-s}{k-s} \binom{n-r}{l-r} S_{s,r} \\
&= \binom{m}{k} \binom{n}{l} - \sum_{s=1}^k \sum_{r=1}^l (-1)^{s+r} \binom{m-s}{k-s} \binom{n-r}{l-r} S_{s,r}. \tag{42}
\end{aligned}$$

7 Conclusion

Self-contained proofs of Meyer's (1969) results have been followed by the introduction for the first time of bivariate versions of the Fréchet, Gumbel and Chung inequalities, and demonstration of their monotonicity and convexity properties. The method of proof has been via combinatorial identities, in a relatively simple manner which departs from the usual “events” setting for Bonferroni-type inequalities. This has completed the study of bivariate versions of the known *linear* inequalities studied in Hoppe and Seneta (2012). A study of bivariate extension of the univariate *quadratic* inequalities studied in that paper is in progress.

Appendix

We first provide an alternative rationale for using (16) as the “correct” definition of $\overline{S}_{k,l}$.

As noted in our §1, in the univariate “events” setting, the numerator on the right-hand side of Gumbel's inequality (3) for $P(S \geq 1)$ can be written as

$$\binom{m}{k} - \overline{S}_k = \sum_{i \in I_{k,m}} P(A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}).$$

This is also the numerator of the Gumbel-type upper bound for $P(S \geq s)$, from Hoppe and Seneta (2012), §5.2. It is then plausible that in the bivariate generalization, in the “events” situation, the numerator of the Gumbel bound for $P(S \geq s, T \geq t)$

which is just notation for $P(S \geq s \cap T \geq t)$ should have numerator

$$\binom{m}{k} \binom{n}{l} - \bar{S}_{k,l} = \sum_{i \in I_{k,m}} \sum_{j \in J_{l,n}} P((A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}) \cap (B_{j_1} \cup B_{j_2} \cup \dots \cup B_{j_l})). \quad (43)$$

Taking this as the definition of $\bar{S}_{k,l}$, probability manipulation of the right-hand side of (43) gives

$$\bar{S}_{k,l} = \sum_{i \in I_{k,m}} \sum_{j \in J_{l,n}} [P(\bar{A}_{i_1}, \dots, \bar{A}_{i_k}) + P(\bar{B}_{j_1}, \dots, \bar{B}_{j_l}) - P(\bar{A}_{i_1}, \dots, \bar{A}_{i_k}; \bar{B}_{j_1}, \dots, \bar{B}_{j_l})].$$

Thus in order to see that (16) holds, in this “events” setting, we only need to prove that

$$E \binom{m-S}{k} \binom{n-T}{l} = \sum_{i \in I_{k,m}} \sum_{j \in J_{l,n}} P(\bar{A}_{i_1}, \dots, \bar{A}_{i_k}; \bar{B}_{j_1}, \dots, \bar{B}_{j_l}).$$

We can use the method of indicators, as for example in page 292 of Hoppe and Seneta (2012) to do this (Replace M_m with $m - U_m$ and $n - V_n$ respectively there, multiply the two items and then take expectation on both sides).

To conclude this section, we point out that in univariate theory of sets, Boole’s Inequality, using the notation (1):

$$P(S \geq 1) \leq S_1$$

is the first upper Bonferroni Inequality. From (2), the right-hand side equals $E \binom{S}{1} = ES$, to which the right-hand side of Gumbel’s Inequality (3) reduces at $k = 1$. Next, from (15) the first upper Bonferroni Inequality for two sets using notation (5), is:

$$P(S \geq 1, T \geq 1) \leq S_{1,1} = E(ST)$$

from (4).

The proposed inequality (21) for general (S, T) with $k = l = 1$ on the right hand side gives for the bound

$$\frac{E \{ \{ \binom{m}{1} - \binom{m-S}{1} \} \{ \binom{n}{1} - \binom{n-T}{1} \} \}}{\binom{m-1}{1-1} \binom{n-1}{1-1}} = E(ST),$$

which supports, in this sense, (21) as an appropriate bivariate binomial generalization of (3).

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